Inner and Outer Approximating Flowpipes for Delay Differential Equations

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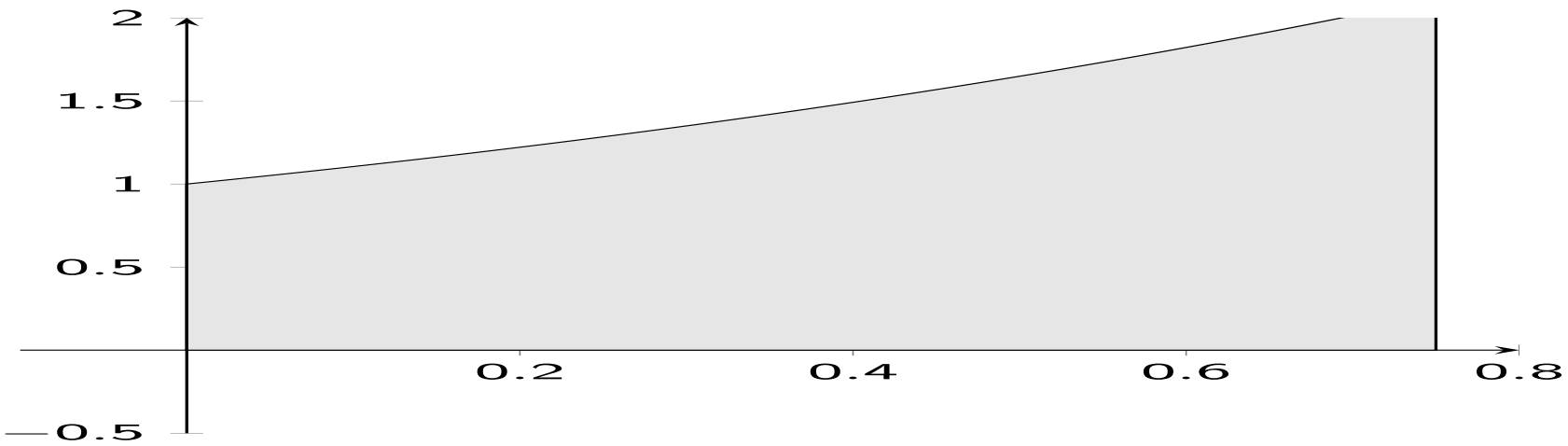
MRIS, March 15, 2018

Motivation: enclosure methods for uncertain dynamical systems

- Computing the reachable sets is central to program analysis, control theory



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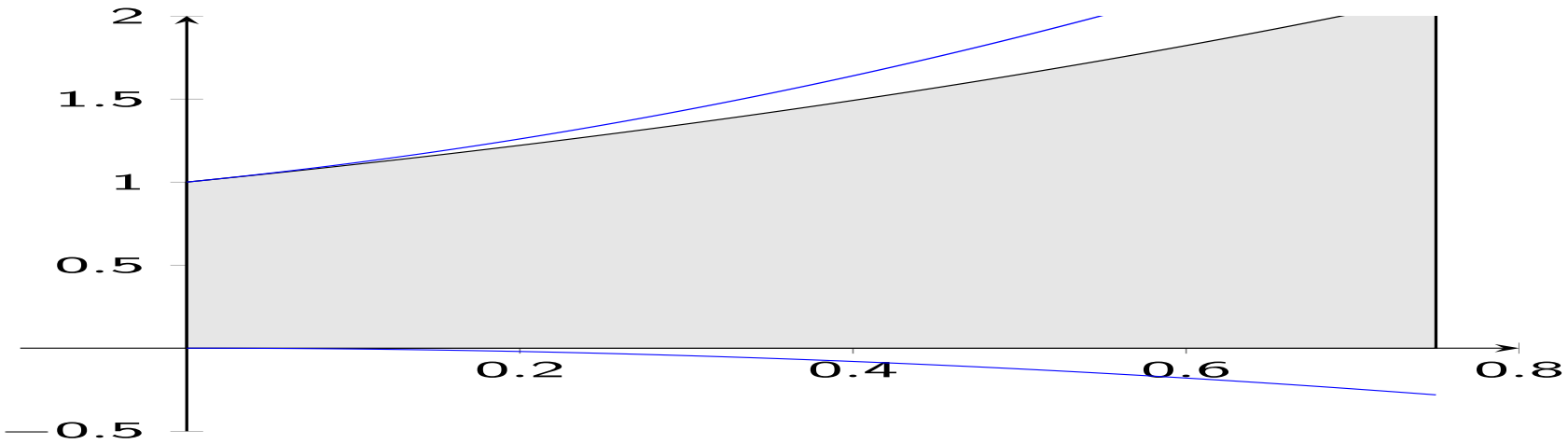

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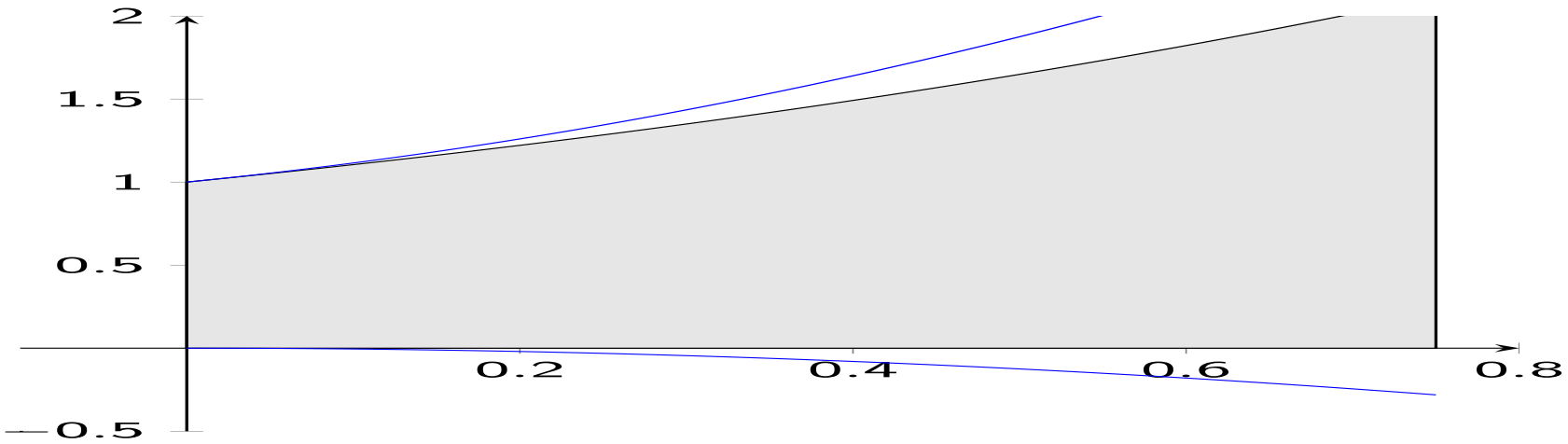

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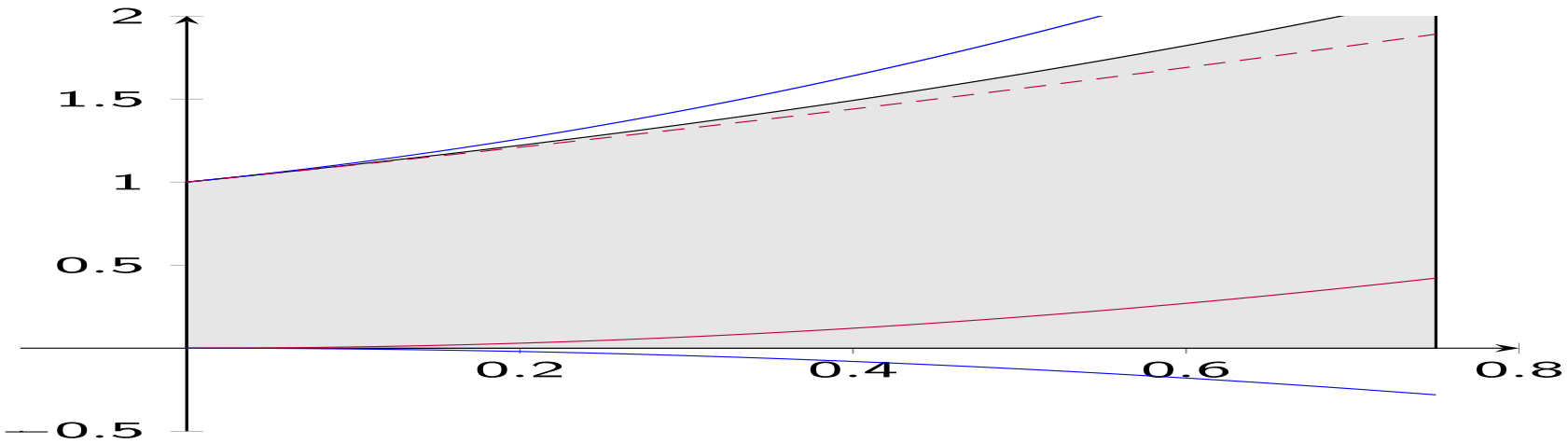

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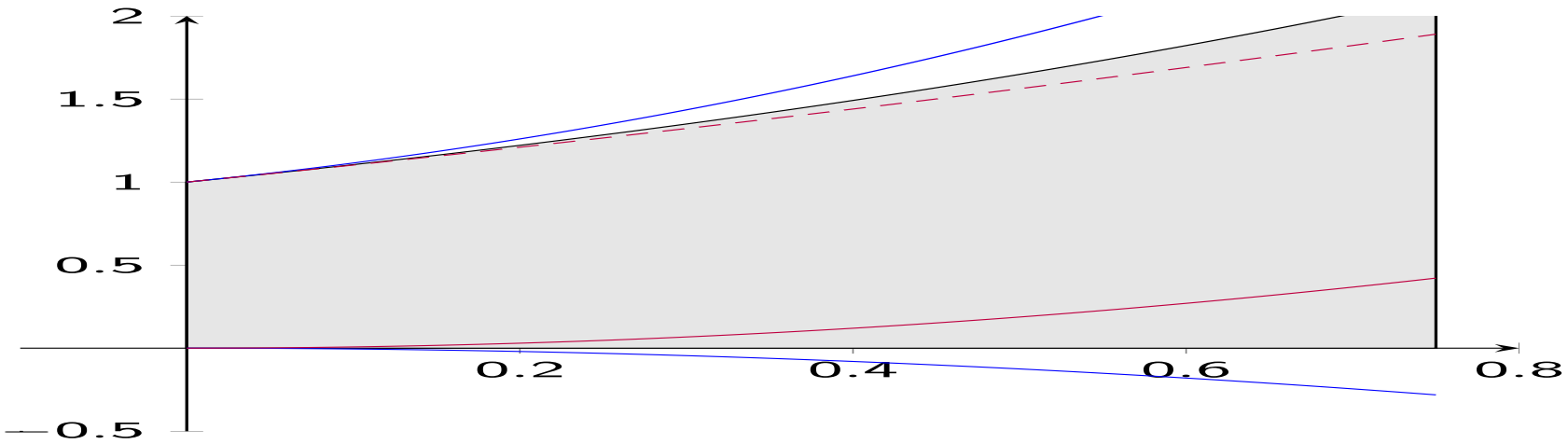
- Here: compute inner-approximated flowpipes = sets of values that are guaranteed to be reached, for some value of the uncertain parameters
  - falsification of safety properties
  - Hausdorff distance between inner and outer tubes gives precision estimates
  - parameter synthesis, verification of new properties (sweep-avoid etc)



And now for delay-differential equations+notion of robust inner-approx!

Intervals, outer and inner approximations

Intervals: closed connected subsets of $\mathbb{R}$, noted $[x] \in \mathbb{I}$; by extension $[x] \in \mathbb{I}^n$ n-dim boxes

For $f : \mathbb{R}^n \to \mathbb{R}^p$, we would like to compute $\text{range}(f, [x]) = \{f(x), x \in [x]\}$.

Outer (or over) approximation

- An *outer approximating extension* of $f : \mathbb{R}^n \to \mathbb{R}$ over intervals is $[f] : \mathbb{I}^n \to \mathbb{I}$ such that
$$\forall [x] \in \mathbb{I}^n, \text{range}(f, [x]) \subseteq [z] = [f]([x])$$
- Natural interval extension: replacing real by interval operations in function f.

Example: the extension of $f(x) = x^2 - x$ on $[2, 3]$ is $[f]([2, 3]) = [2, 3]^2 - [2, 3] = [1, 7]$, and can be interpreted as

$$(\forall x \in [2, 3])\,(\exists z \in [1, 7])\,(f(x) = z).$$

Inner (or under) approximation

An interval inner approximation $[z] \in \mathbb{I}$ satisfies $[z] \subseteq \text{range}(f, [x])$ of the range of f over $[x]$, can be interpreted as

$$(\forall z \in [z])\,(\exists x \in [x])\,(f(x) = z).$$

Delay-differential equations

Form we are considering

$$\begin{aligned} \dot{z}(t) &= f(z(t), z(t-\tau), \beta) \quad \text{if } t \in [t_0 + \tau, T] \\ z(t) &= z_0(t, \beta) \quad \text{if } t \in [t_0, t_0 + \tau] \end{aligned}$$

(slightly less general in the presentation than it could be, e.g. multiple delays, variable delays etc.)

Example : autonomous vehicle

Basic PD-controller for a self-driving car, controlling the car's position x and velocity v ; delay for getting the distance from the sensor.

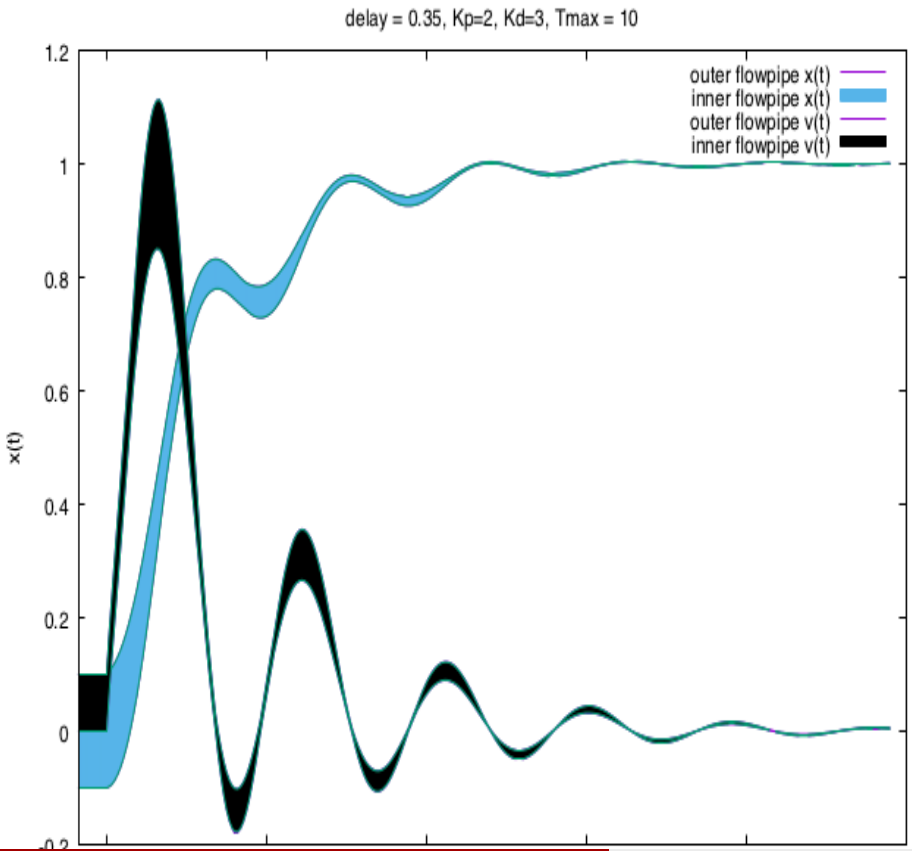
$$\begin{cases} x'(t) = v(t) \\ v'(t) = -K_p\big(x(t-\tau) - p_r\big) - K_d\, v(t-\tau) \end{cases}$$

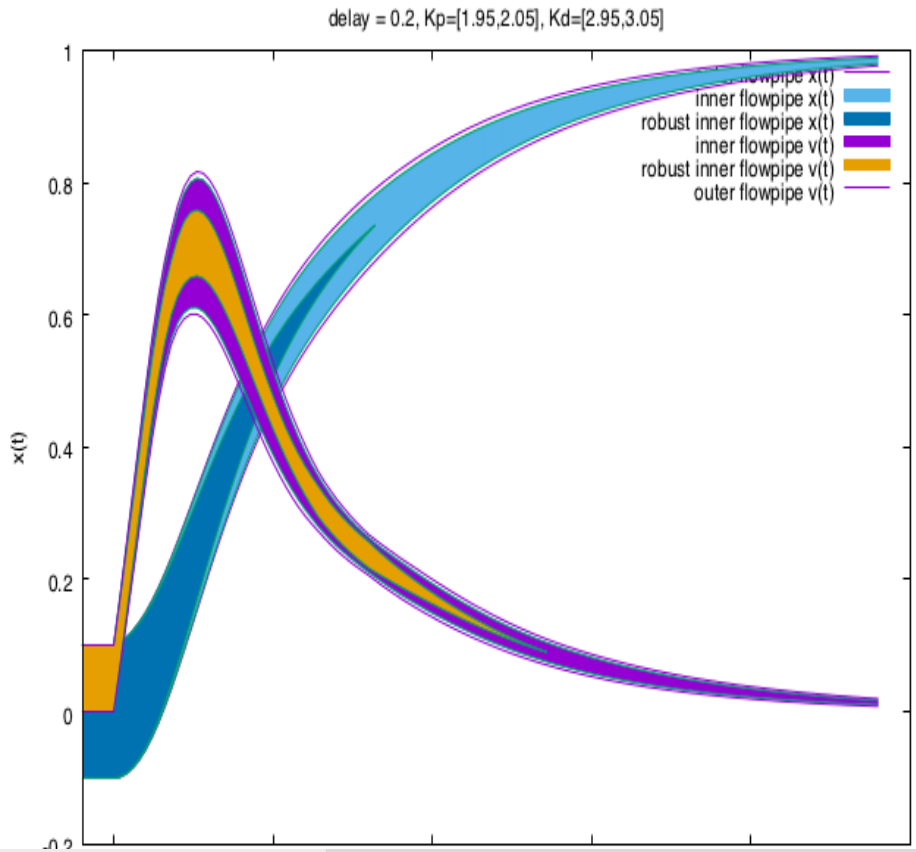
For the initial state, $(x, v) \in [-0.1, 0.1] \times [0, 0.1]$ on the time interval $[-\tau, 0]$.

Simple motivating example : autonomous vehicle

Delays can induce instabilities or weird behaviors!

Choosing $K_p = 2$ and $K_d = 3$ guarantees the asymptotic stability of the controlled system when there is no delay (or small delays).
But even small delays can have a huge impact on the dynamics (left $\tau = 0.35s$, right $\tau = 0.2s$).





A simple running example

Equation

$$\begin{cases} \dot{x}(t) = -x(t) \cdot x(t-\tau) =: f\left(x(t), x(t-\tau), \beta\right) & t \in [0, T] \\ x(t) = x_0(t, \beta) = (1+\beta t)^2 & t \in [-\tau, 0] \end{cases}$$

Simple to solve analytically here, at least for small times

- On $t \in [0, \tau]$ the solution of the DDE is solution of the ODE

$$\dot{x}(t) = f(x(t), x_0(t-\tau, \beta)) = -x(t)(1+\beta(t-\tau))^2, \ t \in [0, \tau]$$

with initial value $x(0) = x_0(0, \beta) = 1$. It admits the analytical solution

$$x(t) = \exp\left(-\frac{1}{3\beta}\left((1+(t-1)\beta)^3 - (1-\beta)^3\right)\right), \ t \in [0, \tau]$$

- The solution of the DDE on the time interval $[\tau, 2\tau]$ is the solution of the ODE

$$\dot{x}(t) = -x(t)\exp\left(-\frac{1}{3\beta}\left((1+(t-\tau-1)\beta)^3 - (1-\beta)^3\right)\right), \ t \in [\tau, 2\tau]$$

with initial value $x(\tau)$ Analytical solution using the transcendental lower γ function.

This is the method of steps for solving DDEs

Principle

- On each time interval $[t_0 + i\tau, t_0 + (i+1)\tau]$, for $i \geq 1$, the function $z(t-\tau)$ is a known history function, already computed as the solution of the DDE on the previous time interval $[t_0 + (i-1)\tau, t_0 + i\tau]$
- Plugging the solution of the previous ODE into the DDE yields a new ODE on the next tile interval

Rest of the talk

- We will use our Taylor model approach (both on the original ODE and on the "variational equations") to derive outer- and inner- approximations of the flow for each ODE derived from the DDE, at each time step - based on our paper HSCC 2017
- The main difficulty will be to represent functions (as initial conditions to each of these ODEs) efficiently, and not just values as for ODEs
- We will also introduce a notion of "robust inner-approximation"

Taylor models for outer-approximated flowpipes of ODEs (Moore, Berz & Makino)

Problem statement (ODE)

- For uncertain dynamical system $\dot{z}(t) = f(z),\ z(t_0) \in [z_0]$ with $f : \mathbb{R}^n \to \mathbb{R}^n$, given a time grid $t_0 < t_1 < \ldots < t_N$, we use Taylor models at order k to outer-approximate the solution $(t, z_0) \mapsto z(t, z_0)$ on each time interval $[t_j, t_{j+1}]$:

$$[z](t, t_j, [z_j]) = [z_j] + \sum_{i=1}^{k-1} \frac{(t - t_j)^i}{i!} f^{[i]}([z_j]) + \frac{(t - t_j)^k}{k!} f^{[k]}([r_{j+1}]),$$

where

- the Taylor coefficients $f^{[i]}$ are the $i-1$th Lie derivative of f along vector field f: defined inductively as follows (can be computed by automatic differentiation)

$$\begin{aligned} f_k^{[1]} &= f_k \\ f_k^{[i+1]} &= \sum_{j=1}^{n} \frac{\partial f_k^{[i]}}{\partial z_j} f_j \end{aligned}$$

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where

- bounding the remainder needs to first compute a (rough) enclosure $[r_{j+1}]$ of solution $z(t, z_0)$ on $[t_j, t_{j+1}]$, classical by Picard iteration: find h_{j+1}, $[r_{j+1}]$ such that

$$[z_j] + [0, h_{j+1}] f([r_{j+1}]) \subseteq [r_{j+1}]$$

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where

- initialization of next iterate $[z_{j+1}] = [z](t_{j+1}, t_j, [z_j])$

Finite representation of solutions of DDEs as Taylor models

On a refined grid!

- Taylor expansions to represent the solution $z(t)$ of the DDE on each time interval $[t_0 + i\tau, t_0 + (i+1)\tau]$,
- For more accuracy, we actually define these expansions piecewise on a finer time grid of fixed time step h.
- Function $z_0(t, \beta)$ on $[t_0, t_0 + \tau]$ represented by $p = \tau/h$ Taylor expansions. The l^{th} such Taylor expansion on $[t_0 + lh, t_0 + (l+1)h]$ with $l \in \{0, \ldots, p-1\}$ is :

$$z_0(t, \beta) = \sum_{i=0}^{k} (t - t_0 - lh)^i z^{[i]}(t_0 + lh, \beta) + (t - t_0 - lh)^{k+1} z^{[k+1]}(\xi_l, \beta),$$

for a $\xi_l \in [t_0 + lh, t_0 + (l+1)h]$.

An abstract Taylor model representation

Several options

- Use a Taylor form in the parameters β for each $z^{[i]}(t_0 + lh, \beta)$, but very costly!
- Good go-between : sub-polyhedric abstraction for Taylor coefficients, in terms of uncertain parameters ("order 1 in parameters, any order in time")
- Here, we use affine forms for the abstraction of parameters : uncertain parameters or inputs $\beta \in \boldsymbol{\beta}$ described by a vector of affine forms over m symbolic variables $\varepsilon_i \in [-1, 1]$: $\boldsymbol{\beta} = \alpha_0 + \sum_{i=1}^{m_j} \alpha_i \varepsilon_i$, where the coefficients α_i are vectors of real numbers.

Example (continued)

- $\boldsymbol{\beta} = [\frac{1}{3}, 1] = \frac{2}{3} + \frac{1}{3}\varepsilon_1$
- Initial conditions $x_0(t, \boldsymbol{\beta})$ is abstracted as a function of the noise symbol ε_1.
- E.g., at $t = -1$, $x_0(-1, \boldsymbol{\beta}) = (1 - \boldsymbol{\beta})^2 = (1 - \frac{2}{3} - \frac{1}{3}\varepsilon_1)^2 = \frac{1}{9}(1 - \varepsilon_1)^2$ abstracted by $\frac{1}{9}(1.5 - 2\varepsilon_1 + 0.5\varepsilon_2)$

Taylor representation

Taylor model in time with zonotopic coefficients

For the initial condition, noting $r_{0j} = [t_0 + jh, t_0 + (j+1)h]$, we write, for all $j = 0, \ldots, p-1$,

- $[z](t) = \sum_{l=0}^{k-1} (t - t_0)^l [z_{0j}]^{[l]} + (t - t_0)^k [\overline{z}_{0j}]^{[k]}, \ t \in r_{0j}$
- where the Taylor coefficients $[z_{0j}]^{[l]} := \frac{[z_0]^{(l)}(t_0 + jh, \boldsymbol{\beta})}{l!}$, $[\overline{z}_{0j}]^{[l]} := \frac{[z_0]^{(l)}(r_{0j}, \boldsymbol{\beta})}{l!}$ can be computed by differentiating the initial solution with respect to t ($[z_0]^{(l)}$ denotes the l-th time derivative), and evaluating the result in affine arithmetic.

Example (continued) - Taylor model of order $k = 2$, step size $h = 1/3$

For the first step $[t_0, t_0 + h] = [-1, -2/3]$:

- $[x_{00}]^{[0]} = [x_0](-1, \boldsymbol{\beta}) = \frac{1}{9}(1.5 - 2\varepsilon_1 + 0.5\varepsilon_2)$
- $[x_{00}]^{[1]} = [\dot{x}_0](-1, \boldsymbol{\beta}) = 2\boldsymbol{\beta}(1 - \boldsymbol{\beta})$
- $[\overline{x_{00}}]^{[2]} = [x_0]^{(2)}(r_I)/2 = [\ddot{x}_0](r_I)/2 = \boldsymbol{\beta}^2$, with $\boldsymbol{\beta} = \frac{2}{3} + \frac{1}{3}\varepsilon_1$

Constructing flowpipes

Method of steps, for Taylor flowpipes

- Plug the Taylor form computed on $[t_0 + (i-1)\tau, t_0 + i\tau]$, into the equation at next time step to get the ODE :

$$\dot{z}(t) = f(z(t), z(t-\tau), \beta), \text{ for } t \in [t_0 + i\tau, t_0 + (i+1)\tau]$$

  where the initial condition $z(t_0 + i\tau)$, and $z(t-\tau)$ for t in $[t_0 + i\tau, t_0 + (i+1)\tau]$ have been previously computed.

- Flowpipes are built using two levels of grids. At each step on the coarser grid with step size τ, we define a new ODE. We build the Taylor models for the solution of this ODE on the finer grid of integration step size $h = \tau/p$.

Step 1

Computing an a priori enclosure

- Classical We iterate the Picard-Lindelöf operator $[F](z) = [z_{ij}] + [t_{ij}, t_{i(j+1)}][f](z, [\overline{z_{i(j-1)}}], \beta)$, with $[\overline{z_{i(j-1)}}]$ the enclosure of the solution over $r_{i(j-1)} = [t_{i(j-1)}, t_{ij}]$
- If this converges, we get the a priori enclosure $[\overline{z_{ij}}]$ on $[t_{ij}, t_{i(j+1)}]$

Building the Taylor model

Taylor expansion on $[t_{ij}, t_{i(j+1)}]$

- $[z](t, t_{ij}, [z_{ij}]) = [z_{ij}] + \sum_{l=1}^{k-1} (t - t_{ij})^l [f_{ij}]^{[l]} + (t - t_{ij})^k [\overline{f_{ij}}]^{[k]}$,
- The Taylor coefficients are defined inductively :

$$\begin{aligned}
[f_{ij}]^{[1]} &= [f]\left([z_{ij}], [z_{(i-1)j}], \beta\right) \\
[f_{1j}]^{[l+1]} &= \frac{1}{l+1}\left(\left[\frac{\partial f^{[l]}}{\partial z}\right][f_{1j}]^{[1]} + [z_{0j}]\,[f_{0j}]^{[1]}\right) \\
[f_{ij}]^{[l+1]} &= \frac{1}{l+1}\left(\left[\frac{\partial f^{[l]}}{\partial z}\right][f_{ij}]^{[1]} + \left[\frac{\partial f^{[l]}}{\partial z^{\tau}}\right]\left[f_{(i-1)j}\right]^{[1]}\right) \quad \text{if } i \geq 2
\end{aligned}$$

- Remainder term : evaluate $[f]$ over the a priori enclosure of the solution on $r_{ij} = [t_{ij}, t_{i(j+1)}]$, e.g. $[\overline{f_{ij}}]^{[1]} = [f]([\overline{z_{ij}}], [\overline{z_{(i-1)j}}])$

Example (continued)

Taylor model of order $k = 2$ on $[t_0 + \tau, t_0 + \tau + h] = [0, 1/3]$

- $[x_{10}] = [x_0](t_{10}, \beta) = [x_0](t_0 + \tau, \beta) = [x_0](0, \beta) = 1.$ and $[f_{10}]^{[1]} = [f]([x_{10}], [x_{00}]) = [f](1, \frac{1}{9}(1.5 - 2\varepsilon_1 + 0.5\varepsilon_2)) = -\frac{1}{9}(1.5 - 2\varepsilon_1 + 0.5\varepsilon_2)$
- $[\overline{f}_{10}]^{[2]} = 0.5\dot{f}(r_{10}, r_{00})$, where r_{i0} for $i = 0, 1$ (with $r_{00} = r_{10} - \tau$) is $[t_{i0}, t_{i1}] = [-1 + i, -1 + i + 1/3]$, and $\dot{f}(t, t - \tau) = \dot{x}(t)x(t - \tau) + x(t)\dot{x}(t - \tau) = f(t, t - \tau)x(t - \tau) + x(t)\dot{x}_0(t - \tau) = -x(t)x(t - \tau)^2 + 2x(t)\beta(1 + \beta t)$. Thus, $[\overline{f}_{10}]^{[2]} = -0.5[x(r_{10})][x(r_{00})]^2 + [x(r_{10})]\beta(1 + \beta r_{10})$
- Enclosures for $x(r_{00})$ and $x(r_{10})$?
- $[x_0](r_{00}) = (1 + \beta r_{00})^2$, evaluated in affine arithmetic
- Evaluating $[x(r_{10})]$ needs the a priori enclosure of the solution on r_{10}. The Picard-Lindelöf operator is $[F](x) = [x_{10}] + [0, \frac{1}{3}][f](x, [x(r_{00})], \beta) = 1 + [0, \frac{1}{3}](1 + \beta r_{00})^2 x$
- We evaluate it in interval for simplicity: $[F](x) = 1 + [0, \frac{1}{3}]\left(1 + [\frac{1}{3}, 1][-1, -\frac{2}{3}]\right)^2 x = 1 + [0, \frac{7^2}{3^5}]x$. Starting with $x_0 = [x_{10}] = 1$, we compute $x_1 = [F](1) = [1, 1 + \frac{7^2}{3^5}]$, $x_2 = [F](x_1) = [1, 1 + \frac{7^2}{3^5} + (\frac{7^2}{3^5})^2]$, converges.

Generalized intervals for outer and inner approximations

Generalized intervals

- Intervals whose bounds are not ordered $K = \{[a, b],\ a \in \mathbb{R}, b \in \mathbb{R}\}$
- Called proper if $a \leq b$, else improper

Definition (Following Goldsztejn et al. 2005)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $[x] \in K^n$, decomposed in $[x]_{\mathcal{A}} \in I^p$ and $[x]_{\mathcal{E}} \in (\text{dual } I)^q$ with $p + q = n$. A generalized interval $[z] \in K$ is $(f, [x])$-interpretable if

$$(\forall x_{\mathcal{A}} \in [x]_{\mathcal{A}})\,(Q_z z \in \text{pro } [z])\,(\exists x_{\mathcal{E}} \in \text{pro } [x]_{\mathcal{E}}), (f(x) = z)$$

where $Q_z = \exists$ if $[z]$ is proper, and $Q_z = \forall$ if $[z]$ is improper.

- When all intervals are proper, we get an outer approximation of range$(f, [x])$

$$(\forall x \in [x])\,(\exists z \in [z])\,(f(x) = z).$$

- When all intervals are improper, we get an inner approximation of range$(f, [x])$

$$(\forall z \in \text{pro } [z])\,(\exists x \in \text{pro } [x])\,(f(x) = z).$$

Kaucher arithmetic [Kaucher 1980] on generalized intervals

Kaucher addition extends addition on classical intervals:
$[x] + [y] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$ and $[x] - [y] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$.

Kaucher multiplication

Let $\mathcal{P} = \{[x] = [\underline{x}, \overline{x}],\ \underline{x} \geqslant 0 \wedge \overline{x} \geqslant 0\}$, $-\mathcal{P} = \{[x] = [\underline{x}, \overline{x}],\ \underline{x} \leqslant 0 \wedge \overline{x} \leqslant 0\}$,
$\mathcal{Z} = \{[x] = [\underline{x}, \overline{x}],\ \underline{x} \leqslant 0 \leqslant \overline{x}\}$, and dual $\mathcal{Z} = \{[x] = [\underline{x}, \overline{x}],\ \underline{x} \geqslant 0 \geqslant \overline{x}\}$.

$[x] \times [y]$	$[y] \in \mathcal{P}$	$\mathcal{Z}$	$-\mathcal{P}$	dual$\mathcal{Z}$
$[x] \in \mathcal{P}$	$[\underline{x}\underline{y}, \overline{x}\overline{y}]$	$[\overline{x}\underline{y}, \overline{x}\overline{y}]$	$[\overline{x}\underline{y}, \underline{x}\overline{y}]$	$[\underline{x}\underline{y}, \underline{x}\overline{y}]$
$\mathcal{Z}$	$[\underline{x}\overline{y}, \overline{x}\overline{y}]$	$[\min(\underline{x}\overline{y}, \overline{x}\underline{y}), \max(\underline{x}\underline{y}, \overline{x}\overline{y})]$	$[\overline{x}\underline{y}, \underline{x}\underline{y}]$	0
$-\mathcal{P}$	$[\underline{x}\overline{y}, \overline{x}\underline{y}]$	$[\underline{x}\overline{y}, \underline{x}\underline{y}]$	$[\overline{x}\overline{y}, \underline{x}\underline{y}]$	$[\overline{x}\overline{y}, \overline{x}\underline{y}]$
dual$\mathcal{Z}$	$[\underline{x}\underline{y}, \overline{x}\underline{y}]$	0	$[\overline{x}\overline{y}, \underline{x}\overline{y}]$	$[\max(\underline{x}\underline{y}, \overline{x}\overline{y}), \min(\underline{x}\overline{y}, \overline{x}\underline{y})]$

Interpretation of Kaucher arithmetic, Goldsztejn et al. 2005

Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by an arithmetic expression *with single occurrences of variables*.
Then for $[x] \in K^n$, $f([x])$, computed using Kaucher arithmetic, is $(f, [x])$-interpretable.

Kaucher arithmetic [Kaucher 1980] on generalized intervals

Kaucher addition extends addition on classical intervals:
$[x] + [y] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$ and $[x] - [y] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$.

Kaucher multiplication

Let $\mathcal{P} = \{[x] = [\underline{x}, \overline{x}],\ \underline{x} \geqslant 0 \wedge \overline{x} \geqslant 0\}$, $-\mathcal{P} = \{[x] = [\underline{x}, \overline{x}],\ \underline{x} \leqslant 0 \wedge \overline{x} \leqslant 0\}$,
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$[x] \times [y]$	$[y] \in \mathcal{P}$	$\mathcal{Z}$	$-\mathcal{P}$	dual$\mathcal{Z}$
$[x] \in \mathcal{P}$	$[\underline{xy}, \overline{xy}]$	$[\overline{x}\underline{y}, \overline{xy}]$	$[\overline{x}\underline{y}, \underline{x}\overline{y}]$	$[\underline{xy}, \underline{x}\overline{y}]$
$\mathcal{Z}$	$[\underline{x}\overline{y}, \overline{xy}]$	$[\min(\underline{x}\overline{y}, \overline{x}\underline{y}), \max(\underline{xy}, \overline{xy})]$	$[\overline{x}\underline{y}, \underline{xy}]$	0
$-\mathcal{P}$	$[\underline{x}\overline{y}, \overline{x}\underline{y}]$	$[\underline{x}\overline{y}, \underline{xy}]$	$[\overline{xy}, \overline{x}\underline{y}]$	$[\overline{xy}, \overline{x}\underline{y}]$
dual$\mathcal{Z}$	$[\underline{xy}, \overline{x}\underline{y}]$	0	$[\overline{xy}, \underline{x}\overline{y}]$	$[\max(\underline{xy}, \overline{xy}), \min(\underline{x}\overline{y}, \overline{x}\underline{y})]$

Interpretation of Kaucher arithmetic, Goldsztejn et al. 2005

Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by an arithmetic expression *with single occurrences of variables*.
Then for $[x] \in K^n$, $f([x])$, computed using Kaucher arithmetic, is $(f, [x])$-interpretable.

Example: $[z] = [x] \times [y] = 0$ when $[x] \in \mathcal{Z}$ and $[y] \in$ dual $\mathcal{Z}$

Example: Kaucher multiplication

Example (Interpretation of the Kaucher multiplication in the case $\mathcal{Z} \times$ dual $\mathcal{Z}$)

$[z] = [x] \times [y] = 0$ when $[x] \in \mathcal{Z} = \{[x],\ \underline{x} \leqslant 0 \leqslant \overline{x}\}$ (e.g. [-5,4]) and $[y] \in$ dual $\mathcal{Z} = \{[x],\ \underline{x} \geqslant 0 \geqslant \overline{x}\}$ (e.g. [1,-1]).

Definition (reminder)

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $[x] \in K^n$, which we can decompose in $[x]_{\mathcal{A}} \in I^p$ and $[x]_{\mathcal{E}} \in (\text{dual } I)^q$ with $p + q = n$. A generalized interval $[z] \in K$ is $(f, [x])$-interpretable if

$$(\forall x_{\mathcal{A}} \in [x]_{\mathcal{A}})\,(Q_z z \in \text{pro } [z])\,(\exists x_{\mathcal{E}} \in \text{pro } [x]_{\mathcal{E}}), (f(x) = z)$$

where $Q_z = \exists$ if $[z]$ is proper, and $Q_z = \forall$ otherwise.

Example: Kaucher multiplication

Example (Interpretation of the Kaucher multiplication in the case $\mathcal{Z} \times$ dual $\mathcal{Z}$)

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Definition (reminder)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $[x] \in I$ and $[y] \in$ (dual I). A generalized interval $[z] \in K$ is $(f, [x] \times [y])$-interpretable if

$$(\forall x \in [x])\,(Q_z z \in \text{pro } [z])\,(\exists y \in [y]), (f(x, y) = x \times y = z)$$

where $Q_z = \exists$ if $[z]$ is proper, and $Q_z = \forall$ otherwise.

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where $Q_z = \exists$ if $[z]$ is proper, and $Q_z = \forall$ otherwise.

Let us suppose $[z]$ improper:

- computing $[z] = [x] \times [y]$ consists in finding $[z]$ such that $\forall x \in [x]$, $\forall z \in$ pro $[z]$, $\exists y \in$ pro $[y]$, $z = x \times y$;
- instanciating the property for $0 \in [x]$, we get $\forall z \in$ pro $[z]$, $(\exists y \in$ pro $[y])$ $z = 0$. Thus $[z]$ is necessarily 0.

Limitations of Kaucher and interval arithmetic

Kaucher arithmetic defines a generalized interval natural extension :

- Interpretable as outer approximation when all intervals are proper (interval arithmetic), but may be insufficiently accurate because of *dependency problem*
- Interpretable as inner approximation when all intervals are improper and f is given by an arithmetic expression *with single occurences of variables*

Example

Let $f(x) = x^2 - x$ that we want to evaluate on $[2, 3]$. Exact range is $\text{range}(f, [2, 3]) = [2, 6]$.

- dependency problem in outer-approximation: accuracy loss $[f]([2, 3]) = [2, 3] * [2, 3] - [2, 3] = [1, 7]$
- single-occurence limitation in inner-approximation: not interpretable $[f]([3, 2])$ computed with Kaucher arithmetic is $[7, 1]$, not $(f, [x])$-interpretable.

A solution: mean-value theorem (and affine arithmetic / zonotopic inductive construction of an outer-approximation)

Solving the single-occurence limitation

Generalized mean-value theorem (Goldsztejn 2005)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable, $[x] \in K^n$, and suppose that for each $i \in \{1, \ldots, n\}$, we can compute $[\boldsymbol{\Delta}_i] \in I$ such that $\left\{ \frac{\partial f}{\partial x_i}(x),\ x \in \text{pro } [x] \right\} \subseteq [\boldsymbol{\Delta}_i]$. Then, for any $\tilde{x} \in \text{pro } [x]$,

$$\tilde{f}([x]) = f(\tilde{x}) + \sum_{i=1}^{n} [\boldsymbol{\Delta}_i]([x_i] - \tilde{x}_i),$$

evaluated with Kaucher interval arithmetic, is $(f, [x])$-interpretable. In particular,

- if $\tilde{f}(\text{dual pro } [x])$, computed with Kaucher arithmetic, is improper, then pro $\tilde{f}(\text{dual pro } [x])$ is an inner approximation of $\{f(x),\ x \in \text{pro } [x]\} = \text{range}(f, [x])$.
- $\tilde{f}(\text{pro } [x])$ is proper and it is an outer approximation of $\text{range}(f, [x])$.

Example (Mean-value theorem for same example $f(x) = x^2 - x$ for $2 \leq x \leq 3$)

$\tilde{f}([x]) = f(2.5) + [f'([2, 3])]([x] - 2.5) = 3.75 + [3, 5]([x] - 2.5)$ is $(f, [x])$-interpretable:

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solves the single-occurence limitation

Inner-approximating flowpipes

Inner-approximation

Given uncertain (constant) parameters $\beta \in \boldsymbol{\beta}$, an inner-approximation at time t of the reachable set, is $]z[(t, \boldsymbol{\beta}) \subseteq z(t, \boldsymbol{\beta})$ such that $(\forall z \in]z[(t, \boldsymbol{\beta}))\,(\exists \beta \in \boldsymbol{\beta})\,(\varphi(t, \beta) = z)$.

Notion of robust inner-approximation

Given uncertain (constant) parameters $\beta = (\beta_{\mathcal{A}}, \beta_{\mathcal{E}}) \in \boldsymbol{\beta}$, an inner-approximation of the reachable set $z(t, \boldsymbol{\beta})$ at time t, robust with respect to $\beta_{\mathcal{A}}$, is a set $]z[_{\mathcal{A}}(t, \boldsymbol{\beta}_{\mathcal{A}}, \boldsymbol{\beta}_{\mathcal{E}})$ such that $(\forall z \in]z[_{\mathcal{A}}(t, \boldsymbol{\beta}_{\mathcal{A}}, \boldsymbol{\beta}_{\mathcal{E}}))\,(\forall \beta_{\mathcal{A}} \in \boldsymbol{\beta}_{\mathcal{A}})\,(\exists \beta_{\mathcal{E}} \in \boldsymbol{\beta}_{\mathcal{E}})\,(\varphi(t, \beta_{\mathcal{A}}, \beta_{\mathcal{E}}) = z)$.

General principle of our algorithm

- Compute an outer-approximation of $z(t, \tilde{\beta})$ and its Jacobian matrix with respect to β at any time t and for some $\tilde{\beta} \in \boldsymbol{\beta}$
- Use the generalized mean-value theorem to derive an inner-approximation ; carefully for robust inner-approximation

Outer-approximation of the Jacobian matrix coefficients

Variational equation

For a DDE with n states and with m parameters

- Jacobian matrix of $z = (z_1, \ldots, z_n)$ with respect to the parameters $\beta = (\beta_1, \ldots, \beta_m)$:

$$J_{ij}(t) = \frac{\partial z_i}{\partial \beta_j}(t)$$

- The entries satisfy the DDE :

$$\dot{J}_{ij}(t) = \sum_{k=1}^{p} \frac{\partial f_i}{\partial z_k}(t) J_{kj}(t) + \sum_{k=1}^{p} \frac{\partial f_i}{\partial z_k^\tau}(t) J_{kj}(t-\tau) + \frac{\partial f_i}{\partial \beta_j}(t)$$

with initial condition $J_{ij}(t) = (J_{ij})_0(t, \beta) = \frac{\partial (z_i)_0}{\partial \beta_j}(t, \beta)$ for $t \in [t_0, t_0 + \tau]$.

Example (continued)

$\dot{J}_{11}(t) = -x(t-\tau)J_{11}(t) - x(t)J_{11}(t-\tau)$ with initial condition $(J_{11})_0(t, \beta) = 2t(1 + \beta t)$.

Computing inner-approximating flowpipes

Algorithm

Compute outer-approximations, on each time interval $[t_{ij}, t_{i(j+1)}]$, of :

1. the solution $z(t, \tilde{\beta})$ with initial function $z_0(t, \tilde{\beta})$ with $\tilde{\beta} \in \boldsymbol{\beta}$
2. the Jacobian $J(t, \beta)$ of the solution, for all $\beta \in \boldsymbol{\beta}$

Exhibiting inner-approximating flowpipes

- for $\beta = (\beta_{\mathcal{A}}, \beta_{\mathcal{E}})$, and note $J_{\mathcal{A}}$ the sub-matrix of the Jacobian corresponding to the partial derivatives with respect to $\beta_{\mathcal{A}}$; denote by $J_{\mathcal{E}}$ the remaining columns
- If for t in $[t_{ij}, t_{i(j+1)}]$, the following is an improper interval

$$]z[_{\mathcal{A}}(t, t_{ij}, \boldsymbol{\beta}_{\mathcal{A}}, \boldsymbol{\beta}_{\mathcal{E}}) = [z](t, t_{ij}, [\tilde{z}_{ij}]) + [J]_{\mathcal{A}}(t, t_{ij}, [J_{ij}])(\boldsymbol{\beta}_{\mathcal{A}} - \tilde{\beta}_{\mathcal{A}}) + [J]_{\mathcal{E}}(t, t_{ij}, [J_{ij}])(\text{dual } \boldsymbol{\beta}_{\mathcal{E}} - \tilde{\beta}_{\mathcal{E}})$$

then (pro $]z[_{\mathcal{A}}(t, t_{ij}, \boldsymbol{\beta}_{\mathcal{A}}, \boldsymbol{\beta}_{\mathcal{E}})$) is an inner-approximation of the reachable set $z(t, \boldsymbol{\beta})$ on $[t_{ij}, t_{i(j+1)}]$ robust to the parameters $\beta_{\mathcal{A}}$

Implementation and Experiments

Protoype in C++

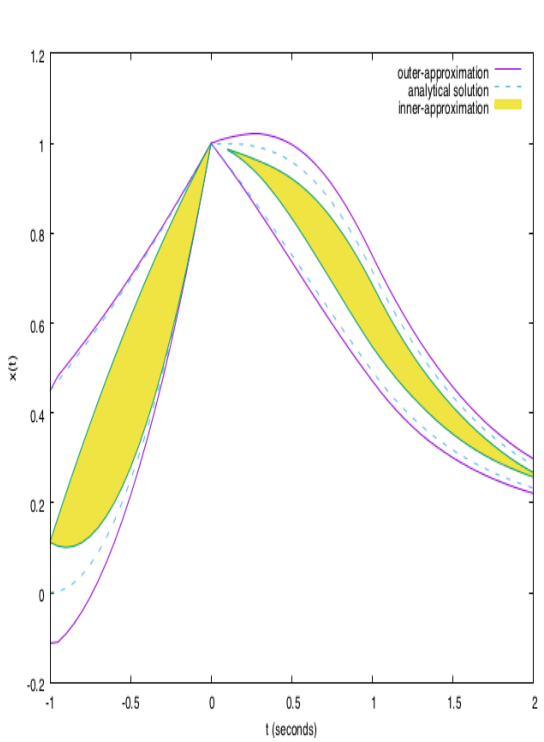
Using :

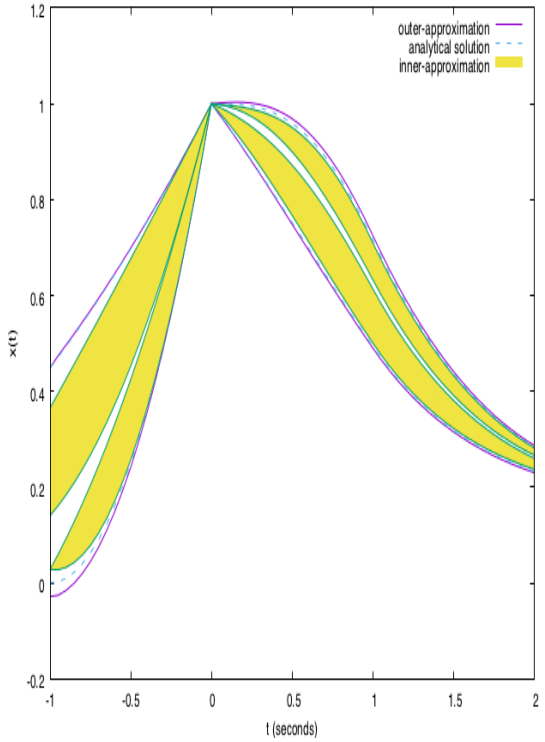
- `FILIB++` C++ library for interval computation
- `FADBAD++` package for automatic differentiation
- and (a slightly modified version of) `aaflib` library for affine arithmetic

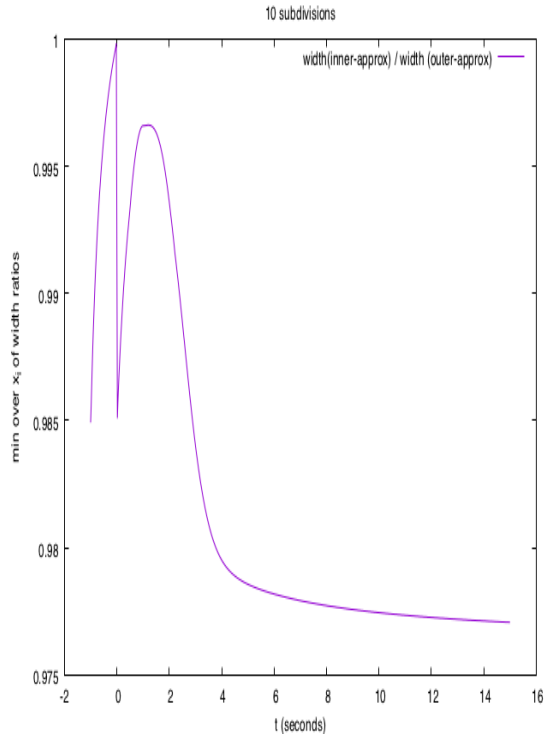
Experiments

Example 1

- Running example, order 2 Taylor models, and integration step size of 0.05
- left : the results until $t = 2$ (obtained in 0.03 seconds) compared to the analytical solution (dashed lines) ; the solid external lines = outer-approximating flowpipe ; the filled region = inner-approximating flowpipe.





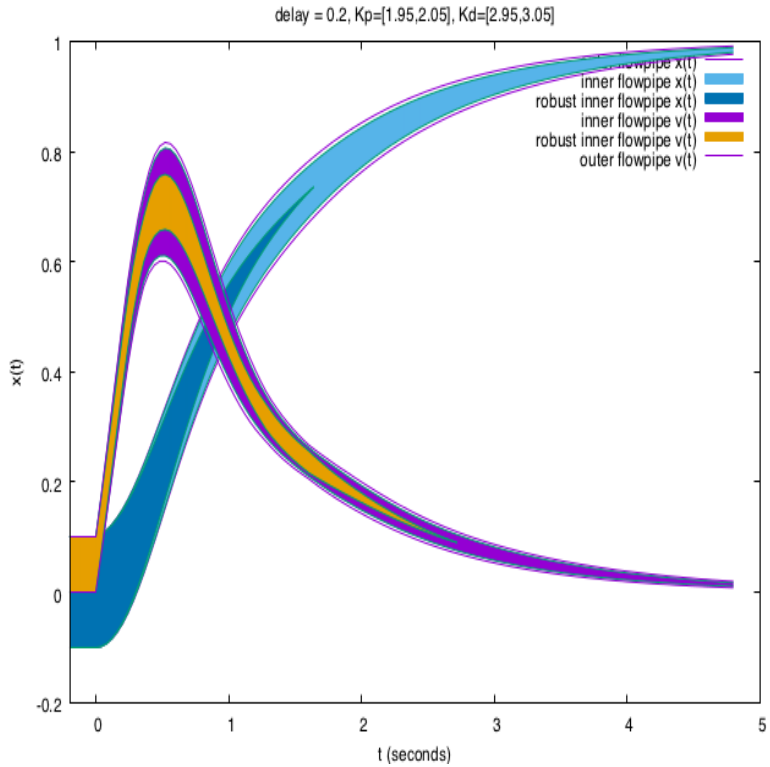


Experiments

Autonomous vehicle

Study of the robustness of the behavior of the system to the PD parameters: constant gains $(K_p, K_d) \in [1.95, 2.05] \times [2.95, 3.05]$. Following results in 0.24s with order 3 Taylor models and time step $= 0.04$

- the outer-approximation : we prove that the velocity never becomes negative
- the inner-approximation
- the robust inner-approximation to the uncertainty in K_p and K_d



A seven-dimensional example

From [Franzle et al. FORMATS 2017]

$$f(x(t), x(t-\tau) = \begin{cases} 1.4x_3(t) - 0.9x_1(t-\tau) \\ 2.5x_5(t) - 1.5x_2(t) \\ 0.6x_7(t) - 0.8x_3(t)x_2(t) \\ 2 - 1.3x_4(t)x_3(t) \\ 0.7x_1(t) - x_4(t)x_5(t) \\ 0.3x_1(t) - 3.1x_6(t) \\ 1.8x_6(t) - 1.5x_7(t)x_2(t)) \end{cases}$$

and the initial function is constant on $[-\tau, 0]$ with values in
$[1.0, 1.2] \times [0.95, 1.15] \times [1.4, 1.6] \times [2.3, 2.5] \times [0.9, 1.1] \times [0.0, 0.2] \times [0.35, 0.55]$

A seven-dimensional example

Results

- Order 2 Taylor models
- Outer-approximation found :
  $([x_1], \ldots, [x_7])(0.1) = ([1.08624, 1.29612], [1.00606, 1.22207], [1.30031, 1.51859], [2.07866, 2.30144], [0.783008, 0.975455], [0.024652, 0.180809], [0.297307, 0.510601])$
- Inner-approximation found : $(]x_1[, \ldots,]x_7[)(0.1) =$
  $([1.08641, 1.29594], [1.00645, 1.22165], [1.30273, 1.51612], [2.08258, 2.29741], [0.785859, 0.972606], [0.0246745, 0.180787], [0.301482, 0.506392])$

Comparison wrt [Franzle et al. FORMATS 2017]

Reachable sets / quality measure γ of the DDE until $t = 0.1$:

	analysis time (sec)	accuracy measure $\gamma(x_1), \ldots, \gamma(x_7)$
our work	0.13	0.998, 0.996, 0.978, 0.964, 0.97, 0.9997, 0.961
Franzle et al.	505	0.575, 0.525, 0.527, 0.543, 0.477, 0.366, 0.523