Rigorous computation of viability kernel

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Outline

Controlled dynamical system and viability kernel

We consider the following controlled nonlinear system:

$$(S) \begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

with $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ the input and $x(t) \in \mathbb{R}^n$ the states, for all $t \geqslant 0$.

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Goal

Given a set of safe states K, find its viability kernel: the set of all initial conditions for which there always exists a trajectory remaining inside K for an indefinite amount of time.

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Viability Kernel

The viability kernel of a set $K \in \mathbb{R}^n$, is defined by:

$$\text{Viab}_S(K) := \{x_0 \in K | (\forall t \in [0,+\infty])(\exists u(t) \in \mathcal{U})(\varphi(x_0,u(t),t) \in K)\},$$

$\varphi(x_0,u(.),.)$ being a solution of (S) starting from x_0 with control input $u(.)$.

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In [Monnet et al., 2015], a two phase framework implemented by interval-based methods:

I. build an initial inner-approximation of the viability kernel (implemented by construction of Lyapunov-like functions)

II. improving the inner-approximation by computing iteratively its capture basin (implemented by guaranteed numerical integration).

Illustration

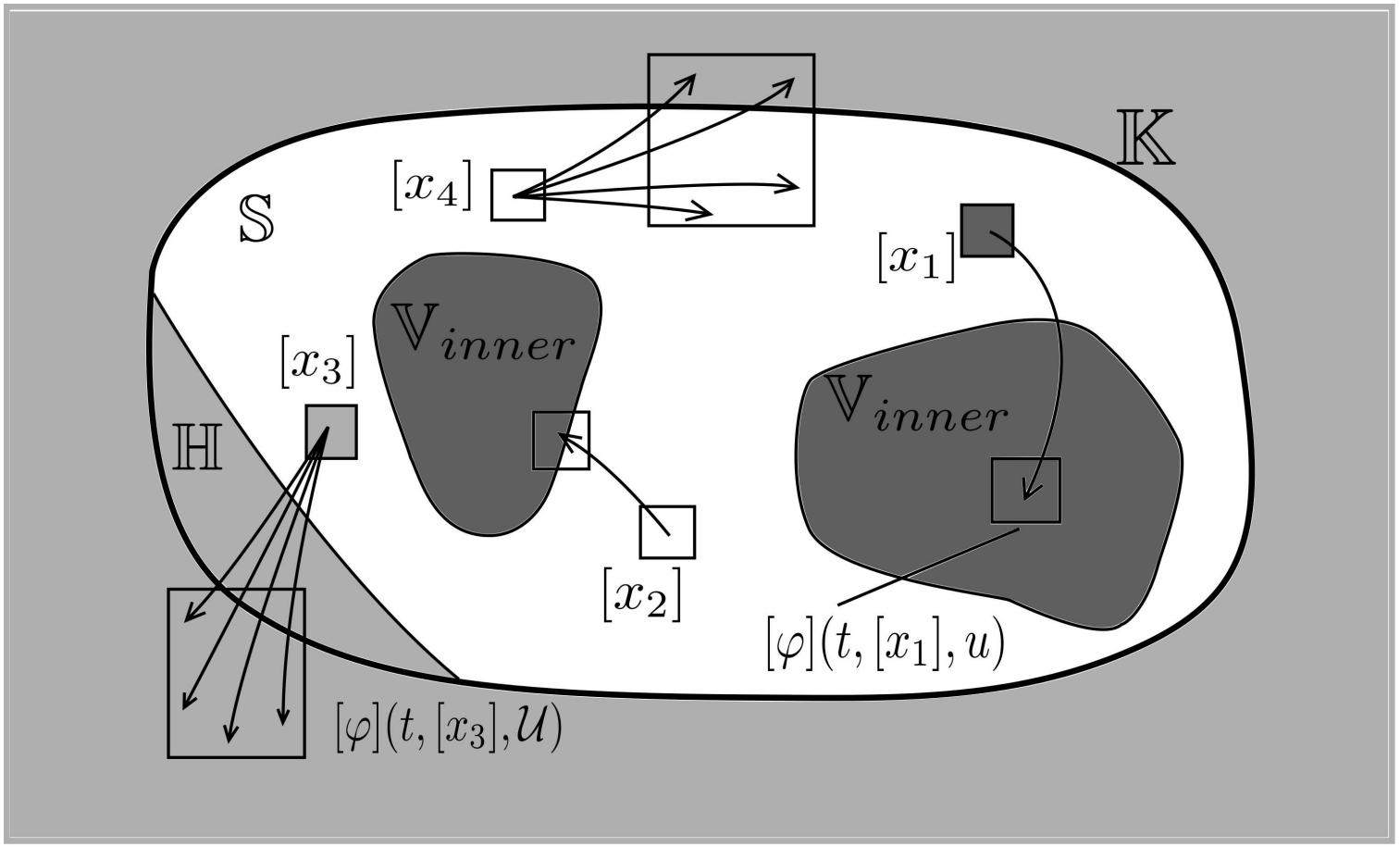
Capture basin

Given a target set T, the capture basin of T within K for a time horizon t_{end} is defined by:

$$\text{Capt}_S^{t_{end}}(K,T) := \left\{ x_0 \in K \;\middle|\; \begin{array}{c} (\exists \tilde{t} \in [0, t_{\text{end}}])(\exists \tilde{u} \in \mathcal{U})(\varphi(x_0, \tilde{u}(\tilde{t}), \tilde{t}) \in T \\ \wedge (\forall t \in [0, \tilde{t}])(\varphi(x_0, \tilde{u}(t), t) \in K)) \end{array} \right\}$$

Main idea: If $T \subseteq \text{Viab}_S(K)$, then $\text{Capt}_S^{t_{end}}(K,T) \subseteq \text{Viab}_S(K)$ for any $t_{\text{end}} \geqslant 0$.

Illustration



Taken from [Monnet et al., 2015]

Interval analysis

Intervals

An interval $[x]$ denotes a connected set of real values, e.g.

$$[x] = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} : \underline{x} \leqslant x \leqslant \overline{x}\}.$$

Vectors of intervals are called boxes. The set of real intervals is denoted $\mathbb{IR}$.

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- safe over-approximation by interval extension of functions (via interval arithmetic)
- proofs of existence and uniqueness of solutions to system of equations e.g. via interval Newton methods
- safe removal of inconsistent values via interval contractors

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Motivation

Taking more time for computing larger initial inner-approximation may reduce significantly the cost of the improvement phase.

First phase: state-of art approach (1)

The core idea is to find a function $h : \mathbb{R}^n \to \mathbb{R}$ of the states satisfying

$$\forall x \in K, \exists u \in \mathcal{U},\ h(x) \neq 0 \vee \langle \nabla_x h(x), f(x,u) \rangle < 0, \tag{1}$$

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Let h be a function satisfying (1), define $\mathcal{H} := \{x \in K |\ h(x) \leqslant 0\}$. Then $\mathcal{H} \subset \mathrm{Viab}_S(K)$, provided the boundary of K and $\mathcal{H}$ does not intersects.

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The last part can be checked in the same manner:

$$\forall x \in K,\ x \notin \delta K \vee h(x) > 0, \tag{2}$$

where δK is the boundary of K.

First phase: original approach (2)

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$$(\tilde{S}_u) \begin{cases} \dot{\tilde{x}} = A\tilde{x} \ , \end{cases}$$

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5. if not found try to find another control such that x^* is stable and compute the corresponding quadratic Lyapunov function,
6. noting by $V(x)$ this Lyapunov function, find the largest r such that $h(x) := V(x) - r$ satisfy (1) (dichotomic search with interval tests).

First phase: brief analysis

Original implementation

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We propose to get back to the original statement. Given a family of parametric template functions $h_p : \mathbb{R}^n \to \mathbb{R}$, find $p \in \mathcal{P} \subset \mathbb{R}^q$ such that:

$$\forall x \in K, \exists u \in \mathcal{U},\ h_p(x) = 0 \vee \langle \nabla_x h(x), f(x,u) \rangle < 0 \tag{3}$$

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Additionally, find a $p \in \mathcal{P}$ verifying (3) that maximizes the volume of $\mathcal{H}_p := \{x \in K |\ h_p(x) \leqslant 0\}$.

How to solve (3)

Barrier function

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Use of contractors [Chabert and Jaulin, 2009] to reduce further the boxes $[p]$ and $[x]$.

CSC-FPS: discussion

Pros and Cons

A branching algorithm on the parameters nesting another branching algorithm on the states.

- Computationally expensive (low dimensions), but it is not necessary to explore all parameters.
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Limitations for our application

I. I. CSC cannot contradict (3), i.e. prove that a p satisfy:

$$\exists x \in K,\ h_p(x) = 0 \wedge \langle \nabla_x h(x), f(x,u) \rangle \geqslant 0.$$

II. Does not consider the "quality" of the returned parameter.
III. Uncontrolled case.

What we propose for I.

I. : Counter-examples

In [Ishii et al., 2012], a Branch & Prune method for solving problems as negation of (3) (uncontrolled case) $\rightarrow$ Use of parametric interval Newton for finding counter-examples to (3).

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I. : Counter-examples

In [Ishii et al., 2012], a Branch & Prune method for solving problems as negation of (3) (uncontrolled case) → Use of parametric interval Newton for finding counter-examples to (3).

Definition

A parametric interval newton operator with respect to a system of equation $h(.,.) = 0$ $(h : \mathbb{R}^{q+n} \to \mathbb{R}^n)^a$ and a parameter box $[p] \in \mathbb{IR}^q$, written $\mathcal{N}_{[p]}(.) : \mathbb{IR}^n \to \mathbb{IR}^n$, is an operator satisfying:

$$\forall [x] \in \mathbb{IR}^n, \mathcal{N}_{[p]}([x]) \subset [x] \implies \forall p \in [p], \exists x \in [x], h(p,x) = 0$$

[a] With fewer equations than n, select some states x_i as parameters.

What we propose for II

II. : Optimizing via Branch & Bound

Embedding the FPS loop into a Branch & Bound optimization algorithm:

- Each iteration selects a preferred box $[p]$ from the branching
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Difficulties

- Definition of a "best" parameter p (which objective function to use for which template)
- Efficient exploration strategy. Following [Neveu et al., 2016], we chose two orderings of the parameter boxes: min LB and max UB. Emphasis on likely feasible parameters ?

What we propose for III.

III. Relaxation by sampling feasible controls

Relaxation of (3): consider a finite set of control functions $\{u_1, u_2, \ldots, u_k\}$, then (3) can be relaxed to:

$$\forall x \in K \; h_p(x) \neq 0 \bigvee_{i=1}^{k} \langle \nabla_x h_p(x), f(x, u_i) \rangle < 0.$$

Choices for the u_i: constant (interval) function, state feedback, ...

Numerical results

Two phase algorithm implemented in Ibex [Chabert and Jaulin, 2009] using DynIbex [2] providing rigorous numerical integration tools using Runge-Kutta methods [Alexandre dit Sandretto and Chapoutot, 2016] for the improvement phase.

[2] `http://perso.ensta-paristech.fr/~chapoutot/dynibex/`

Car on the hill

Consider the system:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -9.81 \sin\left(\frac{1.1\sin(1.2x) - 1.2\sin(1.1x)}{2}\right) - 0.7v + u, \end{cases} \tag{4}$$

with $K = ([-1, 13] \times [-6, 6])^T$ and $u \in [-3, 3]$.

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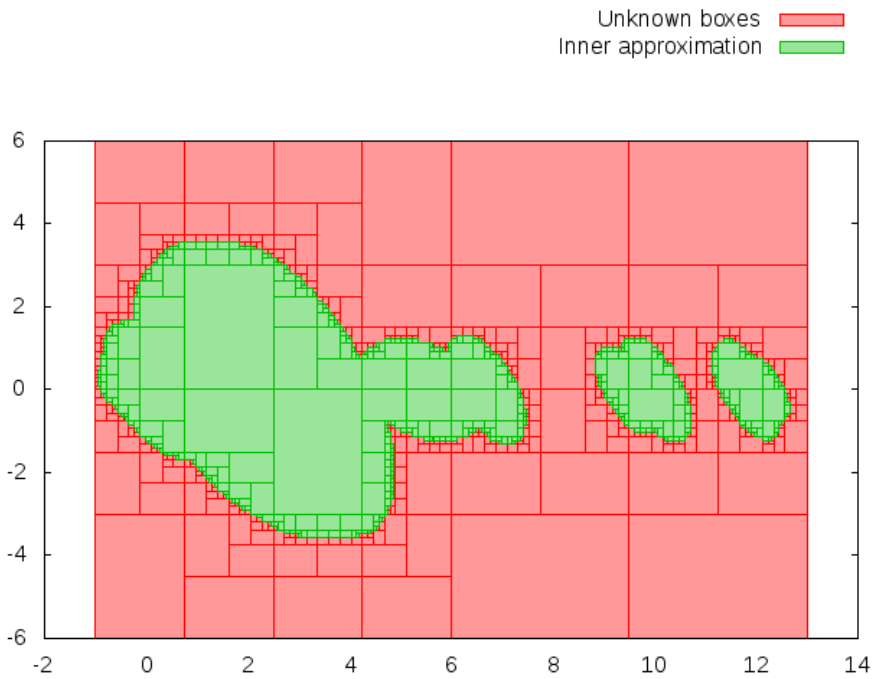
with $K = ([-1, 13] \times [-6, 6])^T$ and $u \in [-3, 3]$.
Considering the equilibrium points $(x^*, 0)$ with control $u = 0$, we use the following 3-parametric template function:

$$h_p(x, v) := p_1(x - x^*)^2 + p_2 v^2 + p_3(x - x^*)v - 1, \tag{5}$$

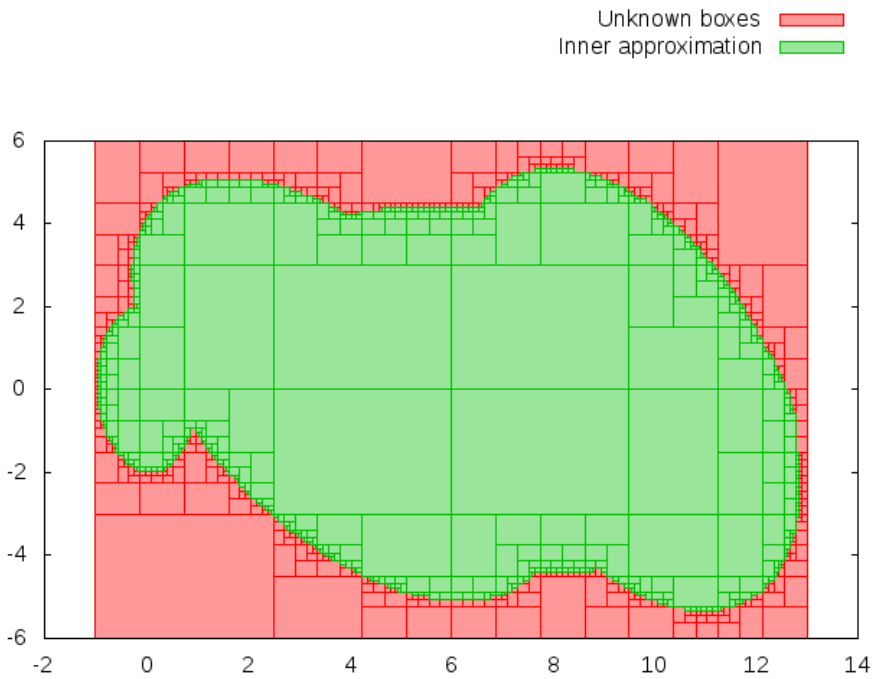
and we impose its positive definiteness, i.e. h_p is a quadratic convex function (additional constraints on p). We minimize the norm of (p_1, p_2).
First phase stopped after 10 seconds for each equilibrium points.

Initial inner-approximation



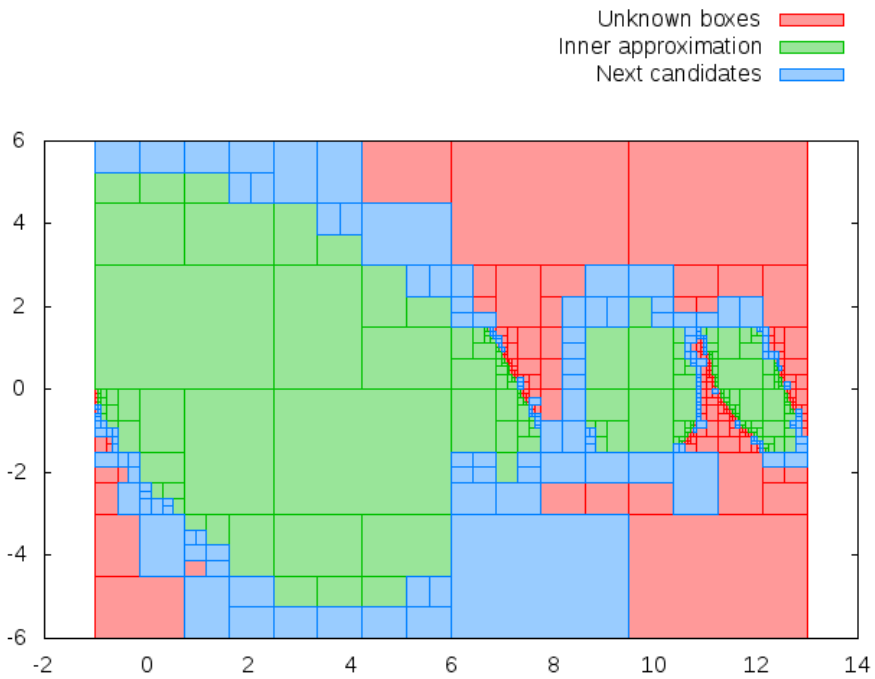
Original method

Initial inner-approximation



New method

After 5 iterations of the improvement procedure



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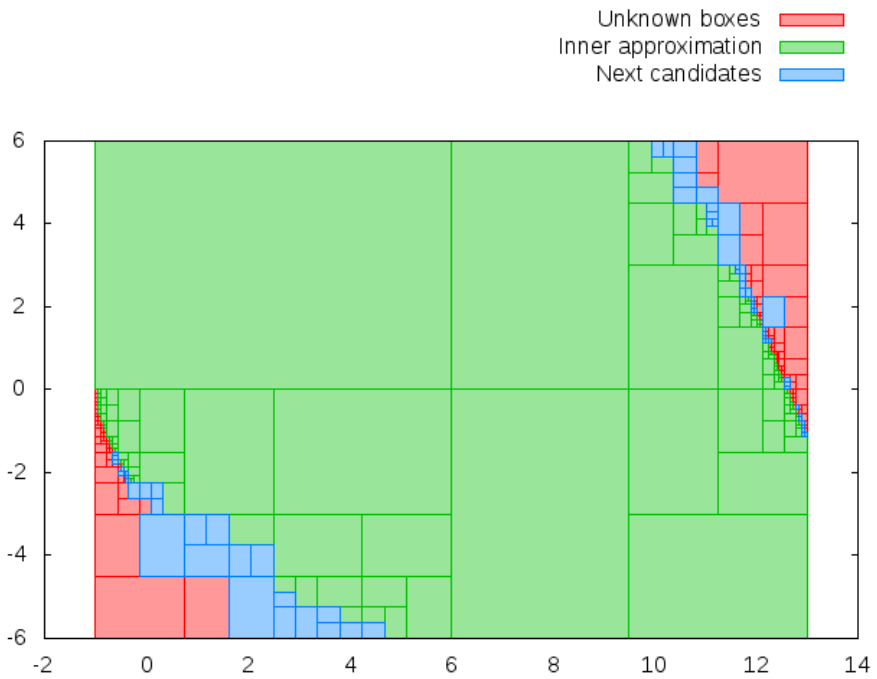
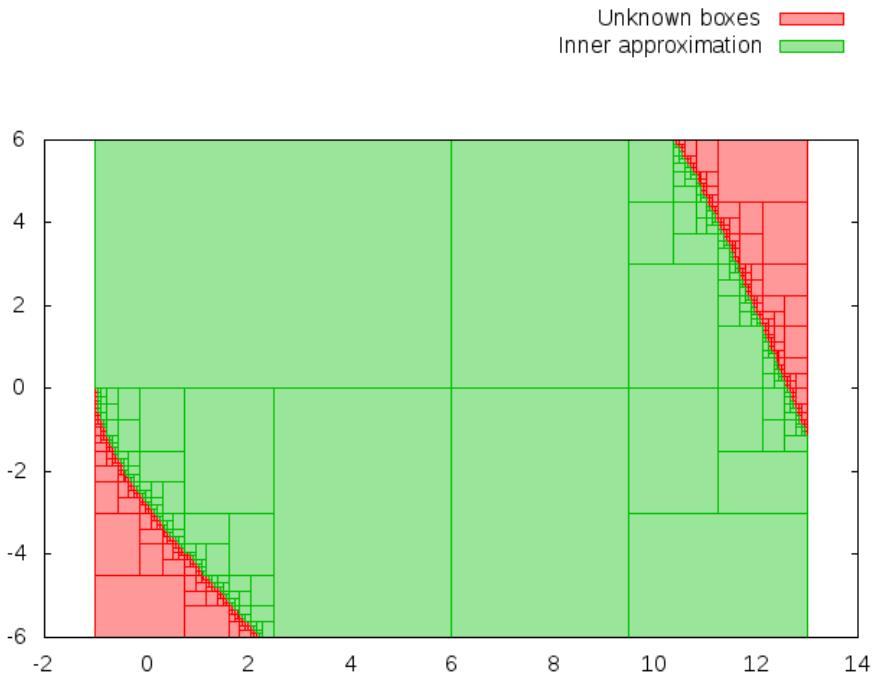


New method

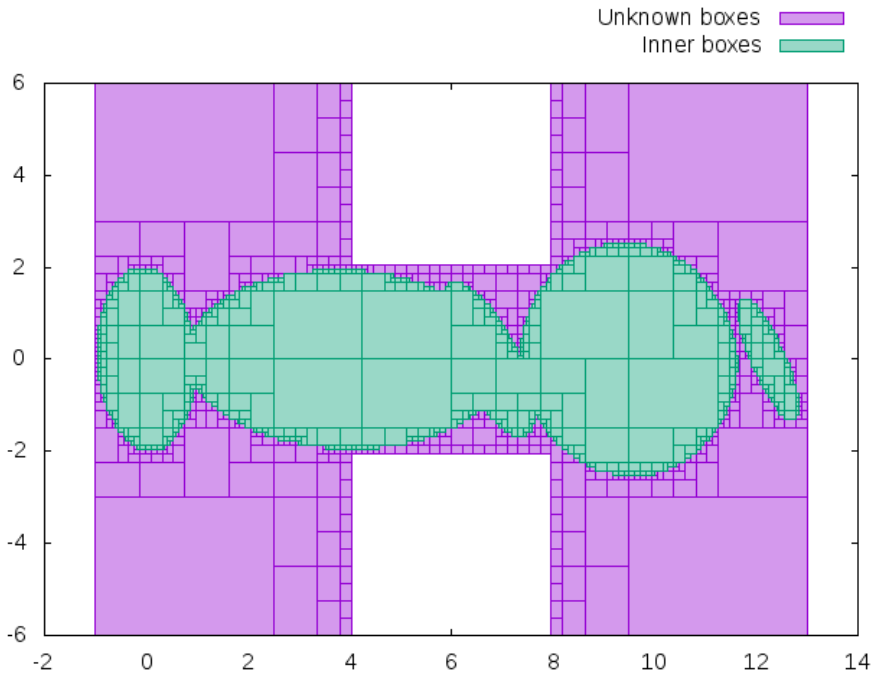
Final inner-approximation



Analysis

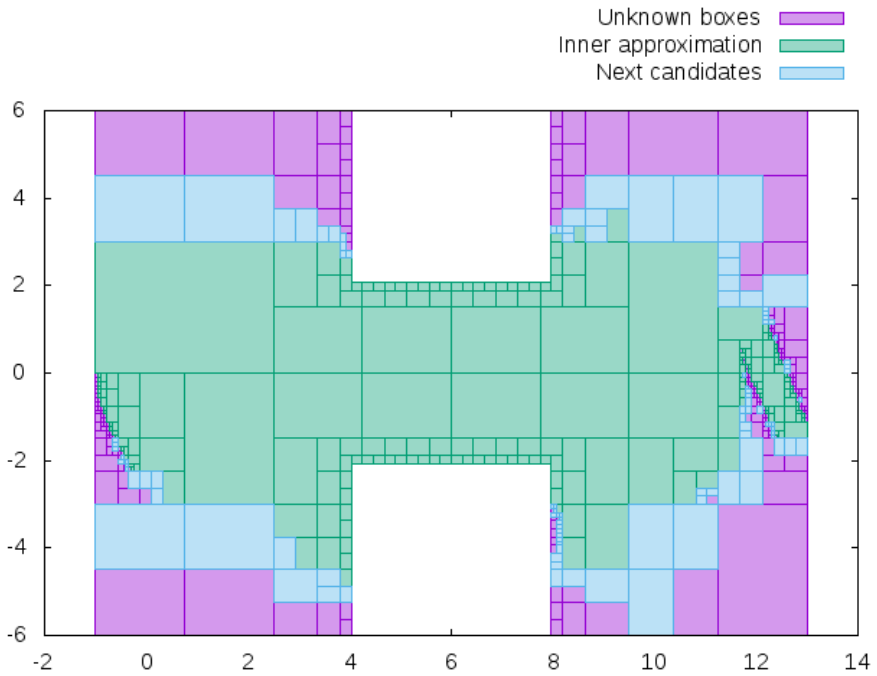
Original method: 27 iterations of improvement procedure, about 360 seconds (3 seconds for the first phase)
New method: 18 iterations of improvement procedure about 175 seconds (34 seconds for the first phase)

Non-convex K



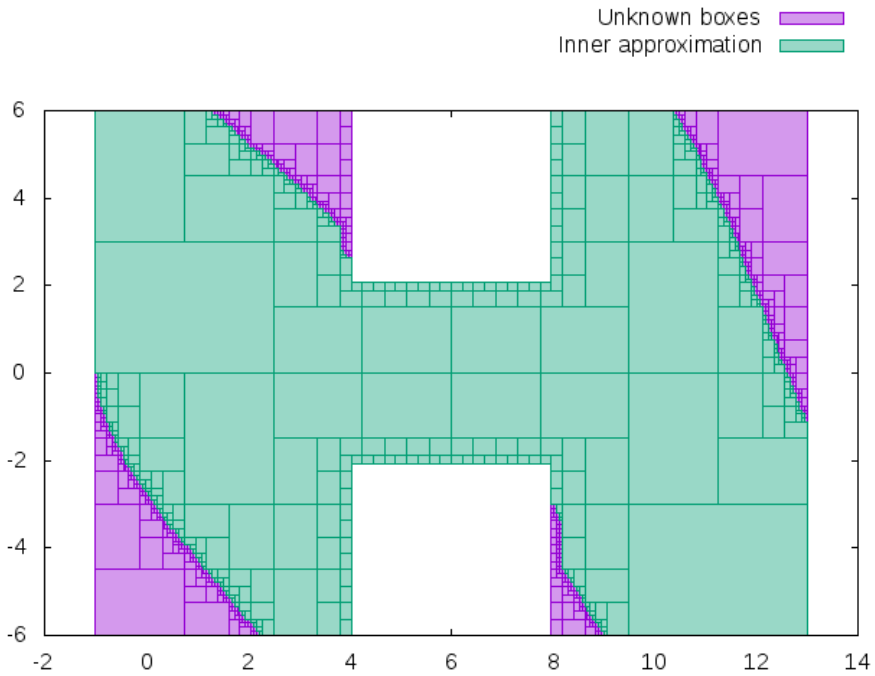
Initial inner-approximation

Non-convex K



After 5 iteration of improvement phase

Non-convex K



Final inner-approximation: 22 iterations, 305 seconds (50s for the first phase)

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- a better method for finding counter-examples or feasible parameters (see e.g. [Hladík and Ratschan, 2014]);
- (meta)heuristic based optimization loop.

Thank you for your attention

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CSC-FPS (1)

FPS loop consists of a branching algorithm on an initial box $[p]$:
Inputs: boxes $[p]$, $[x]$
Main loop (while $Q \neq 0$):

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If never stopped by 5., limit precision is reached then it is not decidable, otherwise no feasible parameter in $[p]$.

CSC-FPS (2)

CSC loop consists of a branching algorihtm on an initial box $[x]$:
Inputs: boxes $[p]$, $[x]$, a point $\tilde{p} \in [p]$
Main loop (while $S \neq 0$)

1. $[x]_0 \leftarrow \mathrm{pop}(S)$;

CSC-FPS (2)

CSC loop consists of a branching algorihtm on an initial box $[x]$:
Inputs: boxes $[p]$, $[x]$, a point $\tilde{p} \in [p]$
Main loop (while $S \neq 0$)

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2. Contract $[x]_0$ with respect to $\tilde{p}$;

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If never stopped by 4., limit precision reached then it is not decidable, otherwise $\tilde{p}$ satisfies the QCSP.